

# Remarks on Cantor's diagonalization proof of 1891

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## Abstract

Remarks on the Cantor's nondenumerability proof of 1891 that the real numbers are noncountable will be given. By the Cantor's diagonal procedure, it is not possible to build numbers that are different from all numbers in a general assumed denumerable sequence of all real numbers. The numbers created on the diagonal of the assumed sequence are not different from the numbers in the assumed denumerable sequence or they do not belong to that sequence.

*Key words:* denumerability, real numbers, countability, cardinal numbers

*MSC:* 11B05

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## 1 Introduction

The first proof that it is impossible to establish a one-to-one correspondence between the natural numbers  $N$  and the real numbers  $\mathfrak{R}$  is older than a century. In December of 1873 Cantor first proved non-denumerability of continuum and that first proof proceeded as follows[1,2,3,4]: Find a closed interval  $I_0$  that fails to contain  $r_0$  then find a closed subinterval  $I_1$  of  $I_0$  such that  $I_1$  misses  $r_1$ ; continue in this manner, obtaining an infinite nested sequence of closed intervals,  $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ , that eventually excludes every one of the  $r_n$ ; now let  $d$  be a point lying in the intersection of all the  $I_n$ 's;  $d$  is a real number different from all of the  $r_n$ .

This proof that no denumerable sequence of elements of an interval (a,b) can contain all elements of (a,b) often is overlooked in favor of the 1891 di-

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agonal argument[5], when reference is made to Cantor's proving the nonenumerability of the continuum. Cantor himself repeated this proof with some modifications[2,3,6,7,8,9,10,11,12,13,14] from 1874 to 1897, and today we have even more variations of this proof given by other authors. However, we have to note that they are in nuce similar; all of them include same modification of the Cantor's idea to derive a contradiction by defining in terms which cannot possibly be in the assumed denumerable sequence. So, in principle, all these proofs do not represent a significant change from Cantor's original idea and we can take them to be the same as the Cantor's proofs.

For the reason of clarity, we will not discuss objections to these proofs that have been raised earlier[15,16,17,18,19,20,21] or the legitimacy of these proofs from intuitionistic points of view [22] and their nonconstructive parts, namely appeal to the Bolzano-Weierstrass theorem[23] and inclusion of impredicative methods[24]. We will focus to show what is in principle wrong with the Cantor's 1891 proof and consequently all other similar proofs.

## 2 Remarks on Cantor's 1891 diagonal proof of the nondenumerability of real numbers

### Theorem 1

By the Cantor's diagonal procedure, it is not possible to build numbers that are different from all numbers in a general assumed denumerable sequence of all real numbers or created real numbers do not belong to the considered interval.

### Proof of the theorem 1

Cantor famous method of diagonalization is relaying upon only two elements,  $m$  and  $w$ . With these he considered the collection  $A$  of elements  $E = (x_1, x_2, \dots, x_n, \dots)$ , where each  $x_n$  was either  $m$  or  $w$ . As example:

$$M = (m, m, m, m, \dots),$$

$$W = (w, w, w, w, \dots),$$

$$Emw = (m, w, m, w, \dots).$$

Cantor then asserted that the collection of all such elements  $A$  was nondenumerable.

Let us repeat that proof by considering an open interval of numbers  $(M, W)$ . Cantor first produced a countable listing of elements  $E_\nu$  in terms of the corresponding array (1), where each  $a_{\mu, \nu}$  was either m or w:

$$\begin{aligned}
E_1 &= (a_{11}, a_{12}, \dots, a_{1\nu}, \dots) \\
E_2 &= (a_{21}, a_{22}, \dots, a_{2\nu}, \dots) \\
&\vdots \\
E_\nu &= (a_{\mu 1}, a_{\mu 2}, \dots, a_{\mu \nu}, \dots) \\
&\vdots
\end{aligned} \tag{1}$$

Then Cantor defined a new sequence  $b_1, b_2, \dots, b_\nu, \dots$ , where each  $b_\nu$  was either m or w, determined so that  $b_\nu \neq a_{\nu\nu}$ . By formulating from this sequence of  $b_\nu$  the element  $E_0 = (b_1, b_2, \dots, b_\nu, \dots)$ , it followed that  $E_0 \neq E_\nu$  for any value of the index  $\nu$ .

However, this statements which appears so obvious, that whichever element  $E_\nu$  one might choose to consider, there exists number  $E_0$ , which belongs to sequence (1), and which is always different in  $\nu^{th}$  coordinate, is not correct.

By the Cantor, the number constructed on the diagonal must satisfy that  $b_\nu \neq a_{\nu\nu}$ . But the sequence (1) might be arranged so that all  $a_{\nu\nu} = m$ . Therefore, in that case on the diagonal only one number might be created, which is  $b_\nu = W$ . However, number  $W$  is not inside the interval  $(M, W)$ , so it is not required to be the part of the sequence (1). It is obvious, that in this case Cantor can not establish contradiction, stating that there exists a number that should be part of the sequence (1), but it is not listed in that sequence. This proves that Cantor's theorem is not correct.

### 3 Conclusion

It is impossible by the proposed diagonal procedure to build numbers that are not included in the assumed denumerable set and particularly it is not possible by this way to create an ascending hierarchy, in fact a limitless sequence of transfinite powers.

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